MIT 18.211: COMBINATORIAL ANALYSIS

FELIX GOTTI

Lecture 11: Permutations I

For every $n \in \mathbb{N}$, let S_n denote the set of all permutations of $[n]$. Recall that, by definition, a permutation in S_n can be represented as a linear arrangement of the elements of $[n]$. This representation is often referred to as a word representation. In addition, we have learned in previous lectures how to represent a permutation based on its set of inversions, and we called this representation the inversion table. The primary purpose of this lecture is to introduce a further representation for permutations. This representation, which consists of certain disjoint cycles, is useful in a wide variety of situations.

Recall that a permutation $\pi = w_1 \dots w_n$ in S_n can be considered as a function, namely, the function determined by the assignments $\pi(i) = w_i$ for every $i \in [n]$. We have also seen before that, as functions, permutations are indeed bijections on $[n]$. Conversely, every bijection $w\colon [n] \to [n]$ yields the linear arrangement $w(1) \dots w(n)$, which is a permutation of $[n]$. Under the operation of composition of functions, S_n is a group, and somehow every finite group is inside S_n for some large n. However, we will not delve into the wonderful algebraic structure of S_n as part of this course.

Let $\pi\colon [n] \to [n]$ be a permutation, and fix $k \in [n]$. By the PHP, there exist $i, j \in [\![0, n]\!]$ with $i < j$ such that $\pi^i(k) = \pi^j(k)$, where π^m denote the bijection we obtain by composing π with itself m times (we assume that π^0 is the identity function on $[n]$, which means that $\pi^0(a) = a$ for every $a \in [n]$). Let s be the smallest element in $[\![1, n]\!]$ such that there exists $r \in [\![0, s-1]\!]$ with $\pi^r(k) = \pi^s(k)$. Note that $r = 0$ as, otherwise, the injectivity of π would imply that $\pi^{r-1}(k) = \pi^{s-1}(k)$, contradicting the minimality of s. Hence, for every $k \in [n]$, there exists $s \in [n]$ such that $k, \pi(k), \dots, \pi^{s-1}(k)$ are pairwise distinct and $\pi^s(k) = k$.

Definition 1. For a permutation $\pi\colon [n] \to [n]$ and $k \in [n]$, we call $(k, \pi(k), \dots, \pi^{s-1}(k))$ a *cycle* of π of *length* s provided that $k, \pi(k), \dots, \pi^{s-1}(k)$ are pairwise distinct and $\pi^s(k) = k$.

With notation as in the previous definition, the cycles $(k, \pi(k), \dots, \pi^{s-1}(k))$ and $(\pi^i(k), \pi^{i+1}(k), \dots, \pi^{s-1}(k), k, \dots, \pi^{i-1}(k))$ are considered the same cycle for every $i \in [s-1]$. Therefore each $k \in [n]$ appears in a unique cycle of π, and so we can assign a formal product of disjoint cycles $C_1 \cdots C_\ell$ to π, where every element of $[n]$ belongs to a unique cycle C_i. Two such formal products of cycles are considered the same if one of them can be obtained by permuting the cycles of the other one.

Definition 2. The representation of $\pi \in S_n$ as a formal product of disjoint cycles is called the *disjoint cycle decomposition* of π.

Observe that given a product of disjoint cycles $C_1 \cdots C_\ell$ satisfying that every element of $[n]$ is in one of the cycles C_i, there is a permutation $\pi \in S_n$ whose disjoint cycle decomposition is $C_1 \cdots C_\ell$. We can obtain π as follows: for each k, set $\pi(k) = j$, where j is the first entry in the cycle containing k if k is the last entry and j proceeds k if k is not the last entry.

Example 3. Let us find the disjoint cycle decomposition of the permutation $\pi = 783295146 \in S_9$. First, we identify the cycle containing 1. Since $\pi(1) = 7$ and $\pi(7) = 1$, the desired cycle is $C_1 := (1, 7)$. Now let us identify the cycle containing the smallest element of $[9]$ that is not an entry of C_1, which is 2. Because $\pi(2) = 8$, $\pi(8) = 4$, and $\pi(4) = 2$, the cycle we are looking for is $C_2 := (2, 8, 4)$. Let us proceed to identify the cycle containing 3, which is the smallest element of $[9]$ that is neither an entry of C_1 nor an entry of C_2. Since $\pi(3) = 3$, we get that $C_3 := (3)$. Now we identify the cycle containing 5, the smallest element of $[9]$ that is not an entry of any of the cycles already found. As $\pi(5) = 9$, $\pi(9) = 6$, and $\pi(6) = 5$, the desired cycle is $C_4 := (5, 9, 6)$. Hence the disjoint cycle decomposition of π is $(1, 7)(2, 8, 4)(3)(5, 9, 6)$.

Following standard conventions, we will omit the cycles of length 1 in the disjoint cycle decomposition of any permutation. For instance, if π is the permutation in the previous example, we omit the cycle (3) in its disjoint cycle decomposition, simply writing $\pi = (1, 7)(2, 8, 4)(5, 9, 6)$.

Definition 4. Let $\pi \in S_n$. If for every $i \in [n]$ the disjoint cycle decomposition of π has precisely a_i cycles of length i, then $(a_1, \dots, a_n)$ is called the (*cycle*) *type* of π.

For instance, the cycle type of the permutation π in Example 3 is $(1, 1, 2, 0, 0, 0, 0, 0, 0)$.

Example 5. Let us count the set of permutations of S_9 whose disjoint cycle decompositions have exactly one cycle, that is, whose cycle type is $(0, 0, 0, 0, 0, 0, 0, 0, 1)$. Well, we can choose a linear arrangement $w_1 w_2 \dots w_9$ of the elements of $[9]$ in $9!$ ways, and then we can turn such a linear arrangement into the cycle decomposition $(w_1, w_2, \dots, w_9)$, which consists of precisely one cycle of length 9. However, observe that the 9 rotations of the cycle $(w_1, w_2, \dots, w_9)$ yield the same permutation. Therefore each permutation with cycle type $(0, 0, 0, 0, 0, 0, 0, 0, 1)$ has been counted 9 times. Hence there are $9!/9 = 8!$ permutations of S_9 consisting of exactly one (disjoint) cycle.

Example 6. Now we count the set of permutations of S_7 whose disjoint cycle decompositions consist of two cycles, one of them of length 3, that is, whose cycle type is $(0, 0, 1, 1, 0, 0, 0)$. As in the previous example, we choose a linear arrangement $w_1 \dots w_7$

of $[7]$ in $7!$ ways, and this time we introduce 2 pairs of parentheses to obtain the cycle decomposition $(w_1, w_2, w_3)(w_4, w_5, w_6, w_7)$. Observe that all the 3 rotations of the cycle (w_1, w_2, w_3) yield the same permutation and also all the 4 rotations of the cycle (w_4, w_5, w_6, w_7) yield the same permutation. Therefore we have to compensate the overcounting caused by these rotations, which amounts to dividing by $12 = 3 \cdot 4$. Hence there are $7!/12$ permutations in S_7 whose cycle type is $(0, 0, 1, 1, 0, 0, 0)$.

Keeping the previous two examples in mind, we can establish a formula for the number of permutations of $[n]$ having any prescribed cycle type.

Theorem 7. *Let $a_1, \dots, a_n \in \mathbb{N}_0$ such that $a_1 + 2a_2 + \cdots + na_n = n$. Then the number of permutations with cycle type $(a_1, \dots, a_n)$ is*

$$\frac{n!}{a_1!a_2! \cdots a_n! \cdot 1^{a_1} 2^{a_2} \cdots n^{a_n}}.$$

Proof. Suppose that we have n consecutive blank spaces, and insert $a_1 + \cdots + a_n$ pairs of parenthesis from left to right in n steps as follows. In the i-th step, insert a_i pairs of parentheses in such a way that

(1) the first parenthesis of the first pair is right after the last parenthesis inserted in a previous step (if $i > 1$) and right before the first blank (if $i = 1$),
(2) leave exactly i blanks between each of the a_i pairs of parentheses, and
(3) leave no blank between consecutive pairs of parentheses.

For instance, when $n = 9$ the configuration of blanks and parenthesis corresponding to the cycle type $(1, 2, 0, 1, 0, 0, 0, 0, 0)$ is

$$(\,\_\,)(\,\_\,\_\,)(\,\_\,\_\,)(\,\_\,\_\,\_\,\_\,).$$

Now we can fill the n consecutive blanks by choosing a linear arrangement π of $[n]$ in $n!$ ways. This gives us a permutation whose cycle type is $(a_1, \dots, a_n)$. Now for each $i \in [n]$ there are a_i cycles of length i whose $a_i!$ linear arrangements yield the same permutation; therefore we are overcounting each permutation $a_1!a_2! \cdots a_n!$ times due to this situation. In addition, for each $i \in [n]$ all the i rotations of each of the a_i cycle of length i yield the same permutation; therefore we are overcounting each permutation $1^{a_1} 2^{a_2} \cdots n^{a_n}$ times due to this second situation. Hence we conclude that the number of permutations in S_n with cycle type $(a_1, \dots, a_n)$ is

$$\frac{n!}{a_1!a_2! \cdots a_n! \cdot 1^{a_1} 2^{a_2} \cdots n^{a_n}}.$$

□

Practice Exercises

Exercise 1. [1, Exercise 6.6] *Find a recurrence formula for the number of permutations of S_n whose cube is the identity permutation.*

Exercise 2. [1, Exercise 6.31] *Find the number of permutations of S_{2n} whose largest cycle has length n.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

Department of Mathematics, MIT, Cambridge, MA 02139
Email address: fgotti@mit.edu